(P3) (COMPLEX NUMBERS) (8 – 10 MARKS)

(50% EASY) (50% HARD)

$\boxed{i = \sqrt{-1}}$ $\boxed{i^2 = -1}$

SIMPLIFY

$\sqrt{-36} = \sqrt{36 \times -1} = \sqrt{36} \times \sqrt{-1} = 6i$

$\sqrt{-18} = \sqrt{9 \times 2 \times -1} = \sqrt{9} \times \sqrt{2} \times \sqrt{-1} = 3\sqrt{2}\,i$

Imaginary numbers.

COMPLEX NUMBERS

$$\underset{\text{Complex Number}}{z} = \underbrace{\underbrace{a}_{\text{Real}} + \underbrace{bi}_{\text{Imaginary}}}_{\text{Complex No.}}$$

$$z = \underbrace{3}_{\text{Real}} + \underbrace{4i}_{\text{Imaginary}}$$

SYMBOLS : $Re(z) = 3$

$Im(z) = 4$

$z = 3 + 7i \qquad w = 2 + 5i$

ADD $z + w = (3+7i) + (2+5i) = 5 + 12i$

SUB $z - w = (3+7i) - (2+5i) = 1 + 2i$

MULTIPLY $z \times w = (3+7i)(2+5i)$

$6 + 15i + 14i + 35i^2$

$6 + 29i + 35(-1)$

$6 + 29i - 35$

$\boxed{-29 + 29i}$

DIVIDE $\dfrac{z}{w} = \dfrac{3+7i}{2+5i} \times \dfrac{2-5i}{2-5i}$

Multiply and divide by conjugate of the denominator.

$$\frac{(3+7i)(2-5i)}{(2+5i)(2-5i)}$$

$$\frac{6 - 15i + 14i - 35i^2}{(2)^2 - (5i)^2}$$

$$\frac{6 - i - 35(-1)}{4 - 25i^2}$$

$$\frac{6 + 35 - i}{4 - 25(-1)}$$

Conjugate of a complex number

Change sign of imaginary parts

Symbol: z^*

$z = 3 + 4i$

$z^* = 3 - 4i$

$z = 1 + 2i - \sqrt{3}i$

$z^* = 1 - 2i + \sqrt{3}i$

$$\frac{41 - i}{29}$$ we cannot write a complex number as a single fraction.

$$= \boxed{\frac{41}{29} - \frac{1}{29}i}$$

SQUARE ROOTS OF A COMPLEX NUMBER (5 MARKS)

Q: Find two square roots of $3+4i$ giving your answers in form $a+bi$. (5 marks)

$\sqrt{3+4i} = ??$ (This will also be a complex no.)

$\sqrt{3+4i} = a+bi$ (Now we need a and b)

(a and b must be real)

(STEP 1) SQUARE BOTH SIDES, EXPAND & EQUATE REAL & IMAGINARY PARTS ON BOTH SIDES.

$$\left(\sqrt{3+4i}\right)^2 = (a+bi)^2$$

$$3+4i = a^2 + 2abi + b^2i^2$$

$$3+4i = a^2 + 2abi + b^2(-1)$$

Imaginary parts

$$3 \boxed{+4i} = a^2-b^2 \boxed{+2abi}$$

Real parts

$$a^2-b^2 = 3$$

$$a^2 - \left(\frac{2}{a}\right)^2 = 3$$

$$2abi = 4i$$

$$ab = 2$$

$$\boxed{b = \frac{2}{a}}$$

a

$$a^2 - \frac{4}{a^2} = 3$$

$$\frac{a^4 - 4}{a^2} = 3$$

$$a^4 - 4 = 3a^2$$

$$a^4 - 3a^2 - 4 = 0$$

Substitute $a^2 = x$

$$x^2 - 3x - 4 = 0$$

$$x^2 - 4x + x - 4 = 0$$

$$x(x-4) + 1(x-4) = 0$$

$$(x+1)(x-4) = 0$$

$x = -1$, $x = 4$

$a^2 = -1$ | $a^2 = 4$

$\cancel{a = \pm\sqrt{-1}}$ | $a = \pm\sqrt{4}$

$\cancel{a = \pm i}$ | $a = 2$ or $a = -2$

a & b are real

$b = \frac{2}{a}$ | $b = \frac{2}{-2}$

$b = \frac{2}{2}$

$b = 1$ | $b = -1$

SQUARE ROOT:

$\sqrt{3+4i} = a + bi \longrightarrow 2 + 1i = \boxed{2+i}$

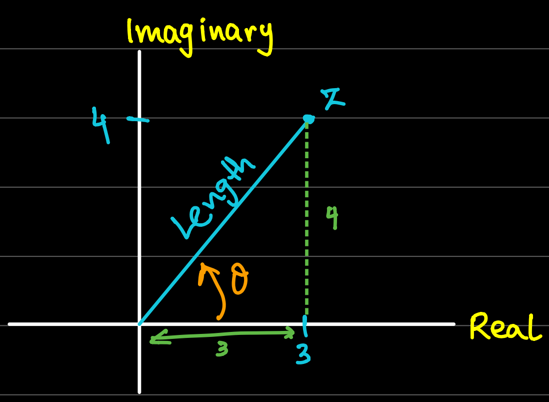
$\longrightarrow -2 + (-1)i = \boxed{-2-i}$

Sunday: S1 (12pm — 2pm) (Representation of Data)

P3 (4pm — 630pm) (complex)

GRAPHICAL REPRESENTATION OF A COMPLEX NUMBER

ARGAND DIAGRAM



$z = a + bi$ (a, b)

$z = 3 + 4i$ $(3, 4)$

MODULUS OF A COMPLEX NUMBER

$z = a + bi$

$R = \sqrt{a^2 + b^2}$ (i) is not part of this formula

$z = 3 + 4i$

$R = \sqrt{3^2 + 4^2}$

$= \sqrt{9 + 16}$

$= \sqrt{25}$

$R = 5$

Wrong $\sqrt{(3)^2 + (4i)^2}$

$\sqrt{9 - 16}$

$\sqrt{-7}$

$R = \sqrt{7}i$

R is a distance.

ARGUMENT (θ) OF A COMPLEX NUMBER

THIS DOES NOT FOLLOW TRIG RULES FOR QUADRANTS

$$\boxed{\begin{array}{c} -180 \leq \arg(\theta) \leq 180 \\ -\pi \leq \arg(\theta) \leq \pi \end{array}}$$

Finding argument is always a two step process:

$$z = a + bi$$

STEP1: Find α

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right)$$

(ignore all +/- signs and i in this)

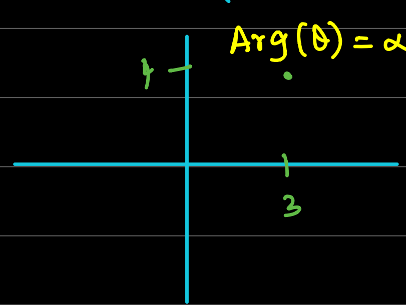
STEP2: Find argument using following Quadrant rules.

$Arg(\theta) = 180 - \alpha$	$Arg(\theta) = \alpha$
$Arg(\theta) = \alpha - 180$	$Arg(\theta) = -\alpha$

(i) $z = 3 + 4i$ (3,4)

(S1) $\alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.13$

(S2)



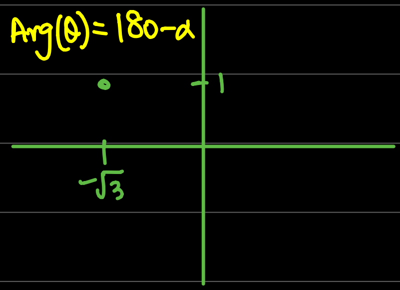
$Arg(\theta) = \alpha = 53.13$

$$\boxed{\theta = 53.13}$$

$\boxed{2}$ $z = -\sqrt{3} + i$

$(-\sqrt{3}, 1)$

(S1) $\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30$

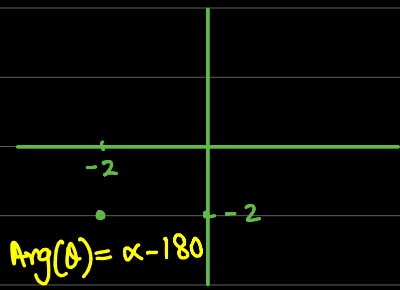


$Arg(\theta) = 180 - \alpha$
$= 180 - 30$
$= 150$

$\boxed{3}$ $z = -2 - 2i$

$(-2, -2)$

(S1) $\alpha = \tan^{-1}\left(\frac{2}{2}\right) = 45$

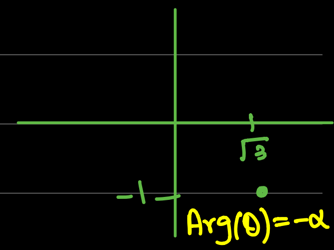


$Arg(\theta) = \alpha - 180$
$= 45 - 180$
$= -135$

$\boxed{4}$ $z = \sqrt{3} - i$

$(\sqrt{3}, -1)$

(S1) $\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
$= 30^\circ$



$Arg(\theta) = -\alpha$
$= -30^\circ$

IF TWO COMPLEX NUMBERS ARE MULTIPLIED THEIR MODULUS ARE ALSO MULTIPLIED AND ARGUMENTS ARE ADDED

IF TWO COMPLEX NUMBERS ARE DIVIDED THEIR MODULUS ARE ALSO DIVIDED AND ARGUMENTS ARE SUBTRACTED

Z_1
Modulus $= R_1$ Argument $= \theta_1$

Z_2
Modulus $= R_2$ Argument $= \theta_2$

$z_1 \times z_2$
Modulus = $R_1 \times R_2$ Argument = $\theta_1 + \theta_2$

$z_1 \div z_2$
Modulus = $R_1 \div R_2$ Argument = $\theta_1 - \theta_2$

5 The complex number $2 + \mathrm{i}$ is denoted by u. Its complex conjugate is denoted by u^*.

$u = 2 + i \quad , \quad u^* = 2 - i$

(ii) Express $\frac{u}{u^*}$ in the form $x + \mathrm{i}y$, where x and y are real. [3]

(iii) By considering the argument of $\frac{u}{u^*}$, or otherwise, prove that

$$\tan^{-1}\left(\tfrac{4}{3}\right) = 2\tan^{-1}\left(\tfrac{1}{2}\right).$$ [2]

(ii) $\frac{u}{u^*} = \frac{2+i}{2-i} \times \frac{2+i}{2+i}$

$$\frac{(2+i)(2+i)}{(2-i)(2+i)}$$

$$\frac{4 + 2i + 2i + i^2}{(2)^2 - (i)^2}$$

$$\frac{4 + 4i + (-1)}{4 - (-1)}$$

$$\frac{3 + 4i}{5}$$

$$\boxed{\frac{u}{u^*} = \frac{3}{5} + \frac{4}{5}i}$$

(iii) $\arg\left(\frac{u}{u^*}\right) = \arg(u) - \arg(u^*)$

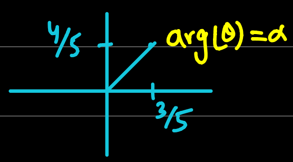
FIRST FIND ARGUMENTS OF ALL THREE

$$\frac{u}{u^*} = \frac{3}{5} + \frac{4}{5}i$$

$$\left(\frac{3}{5}, \frac{4}{5}\right)$$

Step1: $\alpha = \tan^{-1}\left(\dfrac{4/5}{3/5}\right)$

$$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$$

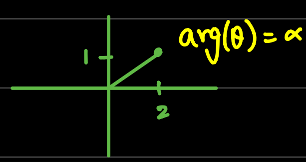


$$\boxed{\arg\left(\frac{u}{u^*}\right) = \tan^{-1}\left(\frac{4}{3}\right)}$$

$$u = 2 + i$$

$$(2, 1)$$

Step1: $\alpha = \tan^{-1}\left(\dfrac{1}{2}\right)$

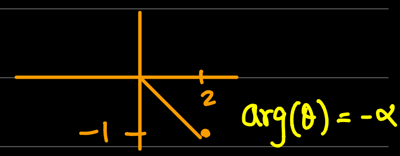


$$\boxed{\arg(u) = \tan^{-1}\left(\frac{1}{2}\right)}$$

$$u^* = 2 - i$$

$$(2, -1)$$

Step1: $\alpha = \tan^{-1}\left(\dfrac{1}{2}\right)$



$$\boxed{\arg(u^*) = -\tan^{-1}\left(\frac{1}{2}\right)}$$

$$\arg\left(\frac{u}{u^*}\right) = \arg(u) - \arg(u^*)$$

$$\tan^{-1}\left(\frac{4}{3}\right) = \left(\tan^{-1}\left(\frac{1}{2}\right)\right) - \left(-\tan^{-1}\left(\frac{1}{2}\right)\right)$$

$$\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\tan^{-1}\left(\frac{4}{3}\right) = 2\tan^{-1}\left(\frac{1}{2}\right)}$$

FORMS OF A COMPLEX NUMBER

RECTANGULAR $z = a + bi$

Usage: Argand Diagrams. $\longrightarrow (a, b)$

POLAR FORM $z = \underbrace{R}_{\text{MODULUS}} (\cos\theta + i\sin\underbrace{\theta}_{\text{Argument}})$

USAGE: Used to convert from Exponential form to rectangular form.

EXPONENTIAL FORM $z = \underbrace{R}_{\text{MODULUS}} e^{i\theta}$ → θ = argument (MUST BE IN RADIANS)

USAGE: TO TAKE HIGHER POWERS OF A COMPLEX NUMBER.

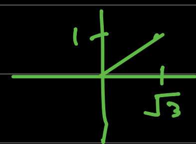
Q: $z = \sqrt{3} + i$

(i) Find modulus and argument of z.

Modulus $= R = \sqrt{(\sqrt{3})^2 + (1)^2}$

$\boxed{R = 2}$

Step 1: $\alpha = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$



$\boxed{\arg(\theta) = \frac{\pi}{6}}$

(ii) Express complex number in form $Re^{i\theta}$ and hence show that z^6 is real.

$z = R\, e^{i\theta}$

$z = 2\, e^{i\left(\frac{\pi}{6}\right)}$

$z^6 = ??$ eg: $(\sqrt{3} + i)^6 \rightarrow$ Binomial (7 term) Too lengthy.

For higher powers of a complex no, use Exp. form.

$z = 2\, e^{i\left(\frac{\pi}{6}\right)}$

$z^6 = \left[2\, e^{i\left(\frac{\pi}{6}\right)}\right]^6$

$= (2)^6 \quad e^{i \frac{\pi}{6} \times 6}$

$z^6 = 64\, e^{i(\pi)}$

NOW GO BACK TO RECTANGULAR FORM USING POLAR FORM.

For z^6, $R = 64$, $\theta = \pi$

$$z^6 = R(\cos\theta + i\sin\theta)$$
$$= 64(\cos\pi + i\sin\pi)$$
$$= 64(-1 + i(0))$$
$$= 64(-1)$$
$$z^6 = -64 \quad (\text{Real})$$

$\boxed{\text{LOCUS}}$

FROM 1 POINT ⟶ CIRCLE

FROM 2 POINTS ⟶ PERPENDICULAR BISECTOR

RESUME STUDY AT 5 pm.